Models of Set Theory II - Winter 2013

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Definition. Suppose that S is an uncountable set and $\kappa > \omega$ is a cardinal. Suppose that $A \subseteq [S]^{<\kappa} = \{X \subseteq S \mid |X| < \kappa\}$ or $A \subseteq [S]^{\kappa} = \{X \subseteq S \mid |X| = \kappa\}$.

(1) A is *unbounded* if for all $x \in [S]^{<\kappa}$ (or $[S]^{\kappa}$), there is some $y \in A$ with $x \subseteq y$.

(2) A is *closed* if for all $\subseteq$*-chains* $(x_\alpha)_{\alpha<\gamma}$ in $[S]^{<\kappa}$ (or $[S]^{\kappa}$), i.e. $(x_\alpha)_{\alpha<\gamma}$ with $x_\alpha \subseteq x_\beta$ for $\alpha < \beta$, if $\bigcup_{\alpha<\gamma} x_\alpha \in [S]^{<\kappa}$ (or $[S]^{\kappa}$) then $\bigcup_{\alpha<\gamma} x_\alpha \in A$.

(3) A is *stationary* if $A \cap C \neq \emptyset$ for every *club* (closed unbounded) $C \subseteq [S]^{<\kappa}$ (or $[S]^{\kappa}$).

Problem 39 (6 Points)**.** Suppose that $\kappa \leq \lambda \leq \mu$ are uncountable regular cardinals. For $Y \subseteq [\mu]^{<\kappa}$, the *projection* of Y to λ is defined as

$$Y_\lambda = \{y \cap \lambda \mid y \in Y\}.$$

For $X \subseteq [\lambda]^{<\kappa}$, the *lifting* of X to μ is defined as

$$X^\mu = \{x \in [\mu]^{<\kappa} \mid x \cap \lambda \in X\}.$$

Show

(a) If S is stationary in $[\mu]^{<\kappa}$, then S_λ is stationary in $[\lambda]^{<\kappa}$.

(b) If C is club in $[\mu]^{<\kappa}$, then C_λ contains a club in $[\lambda]^{<\kappa}$.

(c) If S is stationary in $[\lambda]^{<\kappa}$, then S^μ is stationary in $[\mu]^{<\kappa}$.

(*Hint: Work with clubs of the form C_f for $f\colon [S]^{<\omega} \to [S]^{<\kappa}$ as in the lecture.*)

Problem 40 (6 Points)**.** Suppose that $\kappa \leq \lambda$ are uncountable regular cardinals. If $(X_\alpha)_{\alpha<\lambda}$ is a sequence of subsets of λ, the diagonal intersection is defined as

$$\triangle_{\alpha<\lambda} X_\alpha = \{x \in [\lambda]^{<\kappa} \mid x \in \bigcap_{\alpha \in x} X_\alpha\}.$$

Show

(a) The club filter on $[\lambda]^{<\kappa}$ is closed under diagonal intersections.

(b) If $S \subseteq [\lambda]^{<\kappa}$ is stationary and $f\colon S \to \lambda$ is *regressive*, i.e. $f(x) \in x$ for all $x \in S$, then there is a stationary set $S' \subseteq S$ such that $f \upharpoonright S'$ is constant.

Problem 41 (2 Points)**.** A forcing $(P, \leq, 1)$ *satisfies Axiom A* if there is a collection $(\leq_n)_{n\in\omega}$ of partial orderings of P such that $p \leq_0 q$ implies $p \leq q$, for all n $p \leq_{n+1} q$ implies $p \leq_n q$, and the following conditions hold.

(i) (*Fusion*) If $(p_n)_{n\in\omega}$ is a sequence such that $p_0 \geq_0 p_1 \geq_1 p_2 \ldots$, then there is a condition q such that $q \leq_n p_n$ for all n.

(ii) For every $p \in P$, every n, and every name $\dot{\alpha}$ for an ordinal, there is a condition $q \leq_n p$ and a countable set C such that $q \Vdash_P \dot{\alpha} \in \check{C}$.

Show

(a) Every c.c.c. forcing satisfies Axiom A.
(b) Every ω_1-closed forcing satisfies Axiom A.

Problem 42 (6 Points)**.** Suppose that P is a forcing, $p \in P$, and λ is a cardinal. Consider the following game $G_\lambda(P, p)$ for two players with ω moves. In round n, player I plays a name $\dot{\alpha}_n \in H_\lambda$ for an ordinal and then player II plays a countable set C_n of ordinals. Player II wins if there is a condition $q \leq p$ with

$$q \Vdash_P \forall n \ \dot{\alpha}_n \in \bigcup_{m \in \omega} C_m.$$

A *strategy* for player II is a function which determines the next move of II from the sequence of previous moves. A *winning strategy* for player II is a strategy for II such that II wins for all plays of player I.

Now suppose that P satisfies Axiom A. Show that player II has a winning strategy in the game $G_\lambda(P, p)$ for all $p \in P$.

Please hand in your solutions on Wednesday, January 22 before the lecture.